Summer Bridge Course: Algebra

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Introduction

Field: a field is an abelian group under addition:

1. $a, b \in F \implies a + b \in F$
2. $(a + b) + c \in F \implies a + (b + b) \in F$
3. $\exists 0 \in F$ s.t. $a + 0 = 0 + a, \forall a \in F$
4. $\forall a \in F, \exists - a \in F$ s.t. $a + -a = 0$
5. $a + b = b + a, \forall a, b \in F$
6. $F \setminus \{0\}$ abelian under multiplication
7. $\forall a, b, c \in F, a(b + c) = ab + ac$ and $(a + b)c = ac + bc$

characteristic: Let K be a field and 1_K the multiplicative identity. char(K)=n if n is the smallest natural number s.t. $n1_K = 1_k + \cdots + 1_k = 0_k$. char($K$)=0 if no such n exists.
Gaussian Elimination: use to solve system of equations over $\mathbb{Q}$. Row reduce to get upper triangular matrix, then solve. This can also be thought of as multiplying coefficient matrix by a bunch of elementary matrices.
Elementary Matrix: an $n \times n$ elementary matrix is a matrix of the form $I_n + A$ where A is a matrix with only one non-zero entry (nondiagonal). Basically, all the diagonal entries are 1 and all non-diagonal entries are 0 with exactly one exception.
Permutation Matrix: a matrix with only 0's and 1's such that each row and column contain exactly one 1.
nonsingular: a system of linear equations that has a unique solution.
singular: a system of linear equations is not nonsingular.
consistent: a system of linear equations with at least one solution.
inconsistent: a system of linear equations with no solutions.

Invertibility

Invertible: a matrix $A \in F^{n\times n}$ is invertible if $\exists B \in F^{n\times n}$ s.t. $AB = I_n = BA$.
Theorem: If A is invertible, then the inverse of A is unique.
Theorem: $A_1A_2 \cdots A_m$ invertible $\iff$ both $A_1, A_2, \cdots, A_m$ are invertible.
Theorem: A invertible $\iff$ $Ax = 0$ only has trivial solution.
Theorem: Any product of elementary matrices is invertible.
Theorem: Any permutation matrix is invertible.
Theorem: Consider $A\overrightarrow{x} = \overrightarrow{b}$ and its augmented matrix $[A|\overrightarrow{b}]$. If this can be reduced to $[A'|\overrightarrow{b}']$, then $A'\overrightarrow{x} = \overrightarrow{b}'$ has the same solution set as $A\overrightarrow{x} = \overrightarrow{b}$.
Theorem: An $n \times n$ system of linear equations over F can be reduced to an upper triangular system.
Theorem: A product of upper (lower) triangular matrices is upper (lower) triangular. The diagonal entries are the the product of the respective diagonal entries.
LU Decomposition: Let $A \in F^{n\times n}$. Then $\exists P, L, U \in F^{n\times n}$ s.t.

- L is lower triangular with 1's on diagonal.
- U is upper triangular.
- P is a permutation matrix

$PA = LU$.
How to LU: Row reduce A to an upper triangular matrix (U), keeping track of the row swaps (P) and the scalar multiplications (L) as you go along.
Theorem: Let $U \in F^{n\times n}$ be upper triangular. TFAE:

1. U is invertible.
2. All diagonal entries are nonzero.
3. $U\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution.

This is also true for lower triangular matrices.

Transpose and Symmetric Matrices

Transpose: Let $A = [a_{ij}] \in F^{m\times n}$. The transpose of A is $A^T = [a_{ji}] \in F^{n\times m}$.
Proposition: Let $A, B \in F^{m\times n}$ and $CinF^{n\times k}$.

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AC)^T = C^TA^T$

Theorem: Let $A \in F^{n\times n}$. A is invertible $\iff$ A^T is invertible. Also, if A is invertible, $(A^T)^{-1} = (A^{-1})^T$.
Symmetric: $A \in F^{m\times n}$ is symmetric if $A = A^T$.
Theorem: If $A \in F^{n\times n}$ is invertible and symmetric, then A^{-1} is symmetric.

Vector Spaces

Vector Space: a vector space V over the field F is an abelian group...

1. $\forall v, w \in V, v + w \in V$
2. $\forall u, v, w \in V, u + (v + w) = (u + v) + w$
3. $\exists 0 \in V$ s.t. $0 + v = v + 0, \forall v \in V$
4. $\forall v \in V, \exists w \in V$ s.t.$v + w = w + v = 0$
5. $v + w = w + v, \forall v, w \in V$

equipped with scalar multiplication from F...

1. $a(v + w) = av + aw, \forall a \in F, v, w \in V$
2. $(a + b)v = av + bv, \forall a, b \in F, v \in V$
3. $a(bv) = (ab)v, \forall a, binF, v \in V$
4. $1_Fv = v, \forall v \in V$

Vector Spaces Continued

Subspace: Let V be a vector space over F. a subspace of V, W, is a nonempty subset $W \subseteq V$ s.t. it itself is a vector space.
Theorem: Let $W \subseteq V$ be a subset of V over F. TFAE:

1. W is a subspace of V.
2. $W \neq \emptyset$ and W is closed under addition and scalar multiplication.
3. $W \neq \emptyset$ and W is closed under linear combinations of the form $av + bv$ ($a, b \in F$ and $w_i \in W$).
4. $W \neq 0$ and W is closed under linear combinations of the form $\sum_{i=1}^{k} a_i w_i$ ($a_i \in F$ and $w_i \in W$).

Proposition: $A \in F^{n \times n}$ is invertible iff $N(A) = 0$.

Spanning Sets

Span: Let $v_1, v_2, \cdots, v_n \in V$ and V a vector space over F. Then $\text{Span}_F(v_1, v_2, \cdots, v_n) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n | \alpha_i \in F\}$
Proposition: Let V be a vector space over F. Let $v_1, \cdots, v_n \in V$. Then $\text{Span}_F(v_1, \cdots, v_n)$ is a subspace of V.
Spanning Set: If $v_1, \cdots, v_n \in W \subseteq V$ and $\text{Span}_F(v_1, \cdots, v_n) = W$, we say $v_1, \cdots, v_b$ is a spanning set of W.
Column Space:

$$\text{Let } A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in F^{m \times n}.$$

$$\text{Then } Col(A) = \text{Span}_F(\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix})$$

Row Space: With same $m \times n$ matrix, $Row(A) = \text{Span}_F(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}, \cdots, \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix})$
Theorem: Let $v_1, \cdots, v_n \in V$. Then $\text{Span}_F(v_1, \cdots v_n)$ is a subspace of V containing $v_1, \cdots, v_n$.
Corollary: If $A \in F^{m \times n}$, then $Col(A) \subseteq F^{m \times 1}$ and $Row(A) \subseteq F^{1 \times n}$ are subspaces.

Spanning Sets Continued

Theorem: Let $v_1, \cdots, v_n \in V$ and $W \subseteq V$ a subspace. Then $\text{Span}_F(v_1, \cdots, v_n) \subseteq W \iff v_1, \cdots v_n \in W$.
Notation: Let $v_1, \cdots, v_n \in V$. We write $v_1, \cdots \hat{v}_i, \cdots, v_n = \{v_1, \cdots v_{i-1}, v_{i+1}, \cdots, v_n\}$. (So, we only take out v_i).
Corollary: $\text{Span}_F(v_1, \cdots \hat{v}_i, \cdots, v_n)$ is a subspace of $\text{Span}_F(v_1, \cdots, v_n)$.
Theorem: Let $0 \neq W \subseteq V$ be a subspace. Then $\forall 0 \neq v \in W$, we have $|V| \geq |W| \geq |\text{Span}_F(v)| = |F|$.
Theorem: Let $A \in F^{m \times n}$. Then Col(A)=$\{Ax \in F^m | x \in F^n\} = \{b \in F^m | \exists x \in F^n \text{ s.t. } b = Ax\} = \{b \in F^m |$ the equation $Ax = b$ is consistent $\}$.
Theorem: Let $A \in F^{m \times n}$ and $b \in F^m$.

- The equation $Ax = b$ is consistent iff $b \in \text{Col}(A)$.
- Assume $Ax = b$ is consistent and c a solution to $Ax = b$. Then x is a solution iff $x \in \{c + y | y \in N(A)\}$

Corollary: Let $A \in F^{m \times n}$ and $b \in F^m$. If $Ax = b$ is consistent and $c \in F^n$ is a solution, then the function $\Phi : N(A) \to \{$ solutions of $Ax = n\}$ given by $\Phi(z) = c + z$ is a well-defined bijection.
Corollary: Let $A \in F^{m \times n}$ and $b \in F^m$.

- If $Ax = b$ is consistent and has more then one solution, then the number of solutions is at least $|F|$.
- The number of solutions to $Ax = b$ is 0 (inconsistent), 1, or $\geq |F|$.

Theorem: Let $v_1, \cdots, v_n \in V$. TFAE:

1. $\text{Span}_F(v_1, \cdots, v_n) = \text{Span}_F(v_1, \cdots, \hat{v}_i, \cdots, v_n)$
2. $v_i \in \text{Span}_F(v_1, \cdots, \hat{v}_i, \cdots, v_n)$
3. $\exists a_1, \cdots, a_{i-1}, a_{i+1}, \cdots a_n \in F$ s.t. $v_i = a_1 v_1 + \cdots + a_{i-1} v_{i-1} + a_{i+1} v_{i+1} + \cdots + a_n v_n$
4. $\exists b_1, \cdots, b_n \in F$, not all 0, s.t. $\sum_{j=1}^{n} b_j v_j = 0$
5. $\exists c_1, \cdots, c_n, d_1, \cdots d_n \in F$ with $c_i \neq d_i$ and $\sum_{j=1}^{n} c_j v_j = \sum_{j=1}^{n} d_j v_j$.

Linear Independence

Linearly Independent: Let F be a field and V a vector space over F. Let $S \subseteq V$ be a subset. We say S is linearly independent if any equation of the form $a_1 s_1 + \cdots + a_n s_n = 0 \implies a_i = 0, \forall 1 \leq i \leq n$.
Linearly Dependent: A set is linearly dependent if it is not linearly independent.
Note: If $S = \emptyset$, it is linearly independent.
Note: If you have the zero vector or two of the same vectors in your set, the set is linearly dependent.
Theorem: $\{v_1, \cdots, v_n\}$ is linearly independent $\iff$ any rearrangement is also linearly independent.
Theorem: $\emptyset \neq S \subseteq V$ is linearly dependent $\iff$ $\exists s \in S$ s.t. s is a linear combination of some other elements in S distinct from s.
Theorem: Let F be a field and $v_1, \cdots, v_n \in F^n, n \in \mathbb{N}$. We can form the matris $A = [v_1 | v_2 | \cdots | v_n]$. The list of vectors is linearly independent iff $N(A) = 0$.
Remark: Let $A \in F^{n \times n}$.

- If A is invertible and $b \in F^n$, then $Ax = b$ has a unique solution. ($x = A^{-1} b$)
- TFAE:
  1. A is invertible.
  2. $N(A) = 0$.
  3. $Ax = 0$ has a unique solution.
  4. The columns of A are linearly independent.
  5. The rows of A^T are linearly independent.

Bases

Basis: A basis for V over F is a linearly independent subset $S \subseteq V$ s.t. $\text{Span}_F(S) = V$.
Theorem: The set S is a basis for $V \iff \forall v \in V, \exists! b_1, \cdots, b_n \in F$ s.t. $v = b_1 v_1 + \cdots + b_n v_n$ for some unique $v_1, \cdots, v_n \in S$.
Note: A basis is the maximal linearly independent subset.
Theorem: Let $A \in F^{n \times n}$. TFAE:

1. A is invertible.
2. The columns of A form a basis for F^n.
3. The columns of A are linearly independent.
4. The columns of A span F^n.

Theorem: Let $v_1, \cdots, v_n \in V$ (not all 0). We can reorder these v_i's so that for some $k \leq n$, the list $v_1, \cdots, v_k$ is linearly independent over F and $\text{Span}_F(v_1, \cdots, v_k) = \text{Span}_F(v_1, \cdots, v_n)$. Moreover, every maximal linearly independent sublist of $v_1, \cdots, v_n$ is a basis for $\text{Span}_F(v_1, \cdots, v_n)$.
Theorem: Let $L = (a_1, \cdots, a_n)$ be a linearly independent subset of V. Let $S_0 = (b_1, \cdots, b_m)$ be a spanning set of V, then $n \leq m$.
Corollary: Let V be a vector space and $L = \{u_1, \cdots, u_m\}$ and $M = \{v_1, \cdots, v_n\}$ be bases of V. Then, $n = m$.

Dimension

Convergence Tests for Series

Comparison Tests

1. If $|x_n| \leq c_n, \forall n \geq n_0$, where n_0 is fixed, then $\sum_{k=1}^{\infty} c_k < \infty \implies \sum_{k=1}^{\infty} x_k < \infty$.
2. If $a_k \geq 0, b_k \geq 0$ and $a_k \geq b_k, \forall k \geq n_0$ (n_0 fixed), then $\sum_{k=1}^{\infty} b_k = +\infty \implies \sum_{k=1}^{\infty} a_k = +\infty$.

Limit Comparison Tests: Suppose $a_k \geq 0$ and $b_k \geq 0$. Then,

1. If $\lim_{k\to\infty} \frac{a_k}{b_k} = L, 0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty \iff \sum_{k=1}^{\infty} b_k < \infty$.
2. If $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty} a_k < \infty$.

Integral Test: Let $\{a_k\}$ be a decreasing sequence of nonnegative real numbers ($a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$). Let $f(x) : [1, \infty) \to \mathbb{R}$ and $f(x) \geq 0$ such that f is monotone decreasing and $f(k) = a_k, \forall k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty} a_k < \infty$ iff $\int_1^{\infty} f(x)dx < \infty$.
Root Test: Given $\sum_{k=1}^{\infty} a_k$, let $\alpha = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$.

1. If $\alpha < 1$, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\alpha > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.
3. If $\alpha = 1$, then the test is inconclusive.

Ratio Test: The series $\sum_{k=1}^{\infty} a_k$

1. converges if $\alpha = \overline{\lim}_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$.
2. diverges if $\frac{|a_{n+1}|}{a_n} \geq 1, \forall n \geq n_0$ for some $n_0 \in \mathbb{N}$.

Alternating Series Test: If $\{b_n\} \subseteq \mathbb{R}$ such that

1. $b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots \geq 0$
2. $\lim_{n\to\infty} b_n = 0$

then $\sum (-1)^{k+1} b_k$ converges.
Absolute Convergence: $\sum a_k$ converges absolutely if $\sum |a_k| < \infty$.
Theorem: If $\sum a_k$ converges absolutely, $\sum a_k$ converges.

Important Known Series:

	Geometric	p-Series	$n\log(n)$
	$\sum_{k=1}^{\infty} x^k$	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$
converges	$0 \le x < 1$	$p > 1$	$p > 1$
diverges	$x \ge 1$	$p \le 1$	$p \le 1$

Continuous Functions:

Limit at a point: Given $L \in \mathbb{R}$, $\lim_{x\to a} f(x) = L$ if $\forall \epsilon > 0, \exists \delta(f, \epsilon, a) > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x-a| < \delta$.
Theorem: Let f be a real-valued function defined in some neighborhood $a \in \mathbb{R}$ (including a). Then,

1. f is continuous at a.
   ($\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ if $|x-a| < \delta$).
2. $f(x_n) \to f(a) = L$ whenever $x_n \to a$.

Proof Outline: To show $\lim_{x\to a} f(x) = f(a)$:

1. Do scratch work to find appropriate δ by finding $|f(x) - f(a)| <$ (term involving $|x-a|$) $< \epsilon$.
2. Note that sometimes you need to chose δ to be a minimum of two things to make the inequality true. Be careful!
3. Write out proof and include scratch work.

Right Limit: $\lim_{x\to a^+} f(x) = L^+$ is the right limit if $\forall \epsilon > 0, \exists \delta(f, a, \epsilon) > 0$ such that $|f(x) - L^+| < \epsilon$ if $a < x < a + \delta$.
Left Limit: $\lim_{x\to a^-} f(x) = L^-$ is the left limit if $\forall \epsilon > 0, \exists \delta(f, a, \epsilon) > 0$ such that $|f(x) - L^-| < \epsilon$ if $a - \delta < x < a$.
Continuous at a: f is continuous at a if $f(a^+) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a^-)$
Facts: If f, g are continuous functions at a, then

- $f+g$ is continuous at a.
- fg is continuous at a.
- $\frac{1}{g}$ is continuous at a ($g(x) \neq 0$)

Composition Continuity: $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, and Range$(f) \subseteq B$. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f(x) = g(f(x))$ is continuous at a.

Continuous Functions Continued:

Uniform Continuous: $f : A \subseteq \mathbb{R} \to \mathbb{R}$. f is uniformly continuous on A if $\forall \epsilon > 0, \exists \delta(f, A, \epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x-y| < \delta$.
(Note: δ does NOT depend on a)
Lipschitz Continuous: $f : A \to \mathbb{R}$ is Lipschitz continuous if $\exists M > 0$ such that $|f(x) - f(y)| \le M|x-y|, \forall x, y \in A$.
Fact: Lipschitz $\implies$ uniform $\implies$ continuous
Theorem: If $f : K \to \mathbb{R}$, $K \subseteq \mathbb{R}$ compact, and f continuous on K, then f is uniformly continuous.
Monotone Increasing: f is monotone increasing if $f(x) \le f(y), \forall x < y$. (Strictly if $f(x) < f(y)$)
Monotone Decreasing: f is monotone decreasing if $f(x) \ge f(y), \forall x < y$. (Strictly if $f(x) < f(y)$)
Theorem: If $f : I \to \mathbb{R}$ monotone increasing on I, then $f(p^+)$ and $f(p^-)$ exists for all $p \in I$ and $\sup_{x<p} f(x) = f(p^-) \le f(p) \le f(p^+) = \inf_{x>p} f(x)$.

Sequences and Series of Functions:

Pointwise Limit: Let x_0 be fixed in E. Then $\{f_n(x_0)\} \subseteq \mathbb{R}$. Let $f(x_0) = n_{x_0}$. Let $\{f_n(x_0)\}$ be a sequence of functions such that $f : E \to \mathbb{R}$, then we say f_n converges pointwise on E to f if
$\forall \epsilon > 0, \exists n_0(\epsilon, x_0)$ s.t. $|f_n(x_0) - f(x_0)| < \epsilon, \forall n \ge n_0$.
So, $\lim_{n\to\infty} f_n(x_0) = f(x_0), x_0 \in E$.
Note: Interchangeability of limits, differentiation, and integration is not necessarily true when you just have pointwise continuity. You need something stronger. (Uniform continuity).
Uniform Convergence (Sequence):
a sequence $f_n : E \to \mathbb{R}$ converges uniformly on E to f if $\forall \epsilon > 0, \exists n_0(\epsilon)$ s.t. $|f_n(x) - f(x)| < \epsilon$,
$\forall n \ge n_0, \forall x \in E$.
(Note: n_0 is independent of $x \in E$)
Uniform Convergence (Series):
a series $\sum_{n=0}^{\infty} f_n(x)$; $f_n : E \to \mathbb{R}$ uniformly converges in E iff the sequence of partial sums ($S_k(x) = \sum_{n=0}^{k} f_n(x)$) are uniformly converging to $S(x)$.
Uniformly Cauchy: a sequence of functions $\{f_n(x)\}$; $f_n E \to \mathbb{R}$ is uniformly Cauchy if $\forall \epsilon < 0, \exists n_0(\epsilon)$ s.t $|f_n(x) - f_m(x)| < \epsilon, \forall n, m \ge n_0, \forall x \in E$.

Sequences and Series of Functions Continued:

Sup Norm:

- $\|f\|_\infty = \|f\|_{\text{uniform}} = \|f\|_{\sup} = \sup_{x\in K} |f(x)|$.
- $E = K$ compact $\implies \|f\|_\infty = \max_{x\in K} |f(x)|$.

Sup Norm Convergence: a sequence of functions $\{f_n\}$; $f_n : E \to \mathbb{R}$ converges in the sup norm on E if $\forall \epsilon > 0, \exists n_0(\epsilon)$ such that $\|f_n - f_m\|_\infty = \sup_{x\in E} |f_n(x) - f(x)| < \epsilon, \forall n > n_0$.
Theorem: For a sequence of functions,

Uniform Convergence
$\iff$ Uniformly Cauchy
$\iff$ Sup Norm Convergence

Theorem: $f_n : E \to \mathbb{R}$ and $f_n \in C(E)$.
If f_n converges uniformly to f on E, then $f \in C(E)$.
(Note: To prove this theorem, you use the $\frac{\epsilon}{3}$ trick!)
Corollary: If $\{f_n\} \subseteq (C(E), \|\cdot\|_\infty)$ is Cauchy, then f_n converges uniformly to f on $E \implies f \in C(E) \implies (C(E), \|\cdot\|_\infty)$ is complete.